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GENERALIZED (ANTI) YETTER–DRINFELD MODULES AS COMPONENTS OF A BRAIDED T-CATEGORY

BY

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ABSTRACT

If H is a Hopf algebra with bijective antipode and $\alpha, \beta \in Aut_{Hopf}(H)$, we introduce a category ${}_H\mathcal{YD}^H(\alpha, \beta)$, generalizing both Yetter–Drinfeld modules and anti-Yetter–Drinfeld modules. We construct a braided T-category $\mathcal{YD}(H)$ having all the categories ${}_H\mathcal{YD}^H(\alpha, \beta)$ as components, which, if H is finite dimensional, coincides with the representations of a certain quasitriangular T-coalgebra $DT(H)$ that we construct. We also prove that if (α, β) admits a so-called pair in involution, then ${}_H\mathcal{YD}^H(\alpha, \beta)$ is isomorphic to the category of usual Yetter–Drinfeld modules ${}_H\mathcal{YD}^H$.

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Introduction

Let H be a Hopf algebra with bijective antipode S and $\alpha, \beta \in Aut_{Hopf}(H)$. We introduce the concept of an (α, β)-*Yetter–Drinfeld module*, as being a left H-module right H-comodule M with the following compatibility condition:

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes \beta(h_3)m_{(1)}\alpha(S^{-1}(h_1)).$$

This concept is a generalization of three kinds of objects appeared in the literature. Namely, for $\alpha = \beta = id_H$, we obtain the usual Yetter–Drinfeld modules; for $\alpha = S^2$, $\beta = id_H$, we obtain the so-called anti-Yetter–Drinfeld modules, introduced in [7], [8], [10] as coefficients for the cyclic cohomology of Hopf algebras defined by Connes and Moscovici in [5], [6]; finally, an (id_H, β)-Yetter–Drinfeld module is a generalization of the object H_β defined in [4], which has the property that, for H finite dimensional, the map $\beta \mapsto End(H_\beta)$ gives a group anti-homomorphism from $Aut_{Hopf}(H)$ to the Brauer group of H.

It is natural to expect that (α, β)-Yetter–Drinfeld modules have some properties resembling the ones of the three kinds of objects we mentioned. We will see some of these properties in this paper (others will be given in a subsequent one), namely the ones directed to our main aim here, which is the following: if we denote by ${}_H\mathcal{YD}^H(\alpha, \beta)$ the category of (α, β) -Yetter–Drinfeld modules and define $\mathcal{YD}(H)$ as the disjoint union of all these categories, then we can organize $\mathcal{YD}(H)$ as a braided T-category (or braided crossed group-category, in the original terminology of Turaev, see [16]) over the group $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$ with multiplication $(\alpha, \beta) * (\gamma, \delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$. We also prove that the subcategory $\mathcal{YD}(H)_{fd}$ consisting of finite dimensional objects has left and right dualities, and that, if H is finite dimensional, then $\mathcal{YD}(H)$ coincides with the representations of a certain quasitriangular T-coalgebra $DT(H)$ that we construct.

Our second aim is to prove that, if $\alpha, \beta \in Aut_{Hopf}(H)$ such that there exists a so-called *pair in involution* (f, g) corresponding to (α, β), then ${}_H\mathcal{YD}^H(\alpha, \beta)$ is isomorphic to ${}_H\mathcal{YD}^H$. This result is independent of the theory concerning $\mathcal{YD}(H)$, but we can give it a very short proof using the results obtained during the construction of $\mathcal{YD}(H)$.

1. Preliminaries

We work over a ground field k. All algebras, linear spaces, etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Unless otherwise stated, H will denote a Hopf algebra with bijective antipode S. We will use the versions of Sweedler's sigma notation:

$\Delta(h) = h_1 \otimes h_2$ or $\Delta(h) = h_{(1)} \otimes h_{(2)}$. For unexplained concepts and notation about Hopf algebras we refer to [11], [12], [13], [15]. By $\alpha, \beta, \gamma \ldots$ we will usually denote Hopf automorphisms of H.

Let A be an H-bicomodule algebra, with comodule structures $A \to A \otimes H$, $a \mapsto a_{<0>} \otimes a_{<1>}$ and $A \to H \otimes A$, $a \mapsto a_{[-1]} \otimes a_{[0]}$, and denote, for $a \in A$,

$$a_{\{-1\}} \otimes a_{\{0\}} \otimes a_{\{1\}} = a_{<0>_{[-1]}} \otimes a_{<0>_{[0]}} \otimes a_{<1>} = a_{[-1]} \otimes a_{[0]_{<0>}} \otimes a_{[0]_{<1>}}$$

as an element in $H \otimes A \otimes H$. We can consider the *Yetter–Drinfeld datum* (H, A, H) as in [3] (the second H is regarded as an H-bimodule coalgebra), and the Yetter–Drinfeld category ${}_A\mathcal{YD}(H)^H$, whose objects are k-modules M endowed with a left A-action (denoted by $a \otimes m \mapsto a \cdot m$) and a right H-coaction (denoted by $m \mapsto m_{(0)} \otimes m_{(1)}$) satisfying the equivalent compatibility conditions

$$(a \cdot m)_{(0)} \otimes (a \cdot m)_{(1)} = a_{\{0\}} \cdot m_{(0)} \otimes a_{\{1\}} m_{(1)} S^{-1}(a_{\{-1\}}), \tag{1.1}$$

$$a_{<0>} \cdot m_{(0)} \otimes a_{<1>} m_{(1)} = (a_{[0]} \cdot m)_{(0)} \otimes (a_{[0]} \cdot m)_{(1)} a_{[-1]}, \tag{1.2}$$

for all $a \in A$ and $m \in M$.

Recall now from [9] the construction of the (left) *diagonal crossed product* $H^* \bowtie A$, which is an associative algebra constructed on $H^* \otimes A$, with multiplication given by

$$(p \bowtie a)(q \bowtie b) = p(a_{\{-1\}} \rightharpoonup q \leftharpoonup S^{-1}(a_{\{1\}})) \bowtie a_{\{0\}} b, \tag{1.3}$$

for all $a, b \in A$ and $p, q \in H^*$, and with unit $\varepsilon_H \bowtie 1_A$. Here $\rightharpoonup$ and $\leftharpoonup$ are the regular actions of H on H^* given by $(h \rightharpoonup p)(l) = p(lh)$ and $(p \leftharpoonup h)(l) = p(hl)$ for all $h, l \in H$ and $p \in H^*$.

If H is finite dimensional, we can consider the Drinfeld double $D(H)$, which is a quasitriangular Hopf algebra realized on $H^* \otimes H$; its coalgebra structure is $H^{*cop} \otimes H$ and the algebra structure is just $H^* \bowtie H$, that is

$$(p \bowtie h)(q \bowtie l) = p(h_1 \rightharpoonup q \leftharpoonup S^{-1}(h_3)) \bowtie h_2 l, \tag{1.4}$$

for all $p, q \in H^*$ and $h, l \in H$.

The diagonal crossed product $H^* \bowtie A$ becomes a $D(H)$-bicomodule algebra, with structures

$$H^* \bowtie A \to (H^* \bowtie A) \otimes D(H), \ p \bowtie a \mapsto (p_2 \bowtie a_{<0>}) \otimes (p_1 \otimes a_{<1>}),$$
$$H^* \bowtie A \to D(H) \otimes (H^* \bowtie A), \ p \bowtie a \mapsto (p_2 \otimes a_{[-1]}) \otimes (p_1 \bowtie a_{[0]}),$$

for all $p \in H^*$ and $a \in A$, see [9].

In the case when H is finite dimensional, by results in [1], [3] it follows that the category ${}_A\mathcal{YD}(H)^H$ is isomorphic to the category ${}_{H^*\bowtie A}\mathcal{M}$ of left modules over $H^* \bowtie A$.

2. (α, β)-Yetter–Drinfeld modules

Definition 2.1: Let $\alpha, \beta \in Aut_{Hopf}(H)$. An (α, β)-Yetter–Drinfeld module over H is a vector space M, such that M is a left H-module (with notation $h \otimes m \mapsto h \cdot m$) and a right H-comodule (with notation $M \to M \otimes H$, $m \mapsto m_{(0)} \otimes m_{(1)}$) with the following compatibility condition:

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes \beta(h_3)m_{(1)}\alpha(S^{-1}(h_1)), \tag{2.1}$$

for all $h \in H$ and $m \in M$. We denote by ${}_H\mathcal{YD}^H(\alpha, \beta)$ the category of (α, β)-Yetter–Drinfeld modules, morphisms being the H-linear H-colinear maps.

Remark 2.2: As for usual Yetter–Drinfeld modules, one can see that (2.1) is equivalent to

$$h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}\alpha(h_1). \tag{2.2}$$

Example 2.3: For $\alpha = \beta = id_H$, we have ${}_H\mathcal{YD}^H(id, id) = {}_H\mathcal{YD}^H$, the usual category of (left-right) Yetter–Drinfeld modules.

Example 2.4: For $\alpha = S^2$, $\beta = id_H$, the compatibility condition (2.1) becomes

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S(h_1), \tag{2.3}$$

hence ${}_H\mathcal{YD}^H(S^2, id)$ is the category of anti-Yetter–Drinfeld modules defined in [7], [8], [10].

Example 2.5: For $\beta \in Aut_{Hopf}(H)$, define H_β as in [4], that is $H_\beta = H$, with regular right H-comodule structure and left H-module structure given by $h \cdot h' = \beta(h_2)h'S^{-1}(h_1)$, for all $h, h' \in H$. It was noticed in [4] that H_β satisfies a certain compatibility condition, which actually says that $H_\beta \in {}_H\mathcal{YD}^H(id, \beta)$. More generally, if $\alpha, \beta \in Aut_{Hopf}(H)$, define $H_{\alpha,\beta}$ as follows: $H_{\alpha,\beta} = H$, with regular right H-comodule structure and left H-module structure given by $h \cdot h' = \beta(h_2)h'\alpha(S^{-1}(h_1))$, for $h, h' \in H$. Then one can check that $H_{\alpha,\beta} \in {}_H\mathcal{YD}^H(\alpha, \beta)$.

Example 2.6: Take an integer l and define $\alpha_l = S^{2l} \in Aut_{Hopf}(H)$. The compatibility in ${}_H\mathcal{YD}^H(S^{2l}, id)$ becomes

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S^{2l-1}(h_1). \tag{2.4}$$

An object in ${}_H\mathcal{YD}^H(S^{2l}, id)$ will be called an *$l - \mathcal{YD}$-module*. Hence, a $0 - \mathcal{YD}$-module is a Yetter–Drinfeld module and a $1 - \mathcal{YD}$-module is an anti-Yetter–Drinfeld module. The right–left version of $l - \mathcal{YD}$-modules has been introduced in [14].

Example 2.7: Let α, $\beta \in Aut_{Hopf}(H)$ and assume that there exist an algebra map $f\colon H \to k$ and a group-like element $g \in H$ such that

$$\alpha(h) = g^{-1} f(h_1)\beta(h_2) f(S(h_3))g, \quad \forall h \in H. \tag{2.5}$$

Then one can check that $k \in {}_H\mathcal{YD}^H(\alpha, \beta)$, with structures $h \cdot 1 = f(h)$ and $1 \mapsto 1 \otimes g$. More generally, if V is any vector space, then $V \in {}_H\mathcal{YD}^H(\alpha, \beta)$, with structures $h \cdot v = f(h)v$ and $v \mapsto v \otimes g$, for all $h \in H$ and $v \in V$.

Definition 2.8: If α, $\beta \in Aut_{Hopf}(H)$ such that there exist f, g as in Example 2.7, we will say that (f, g) is a pair in involution corresponding to (α, β) (in analogy with the concept of modular pair in involution due to Connes and Moscovici) and the (α, β)-Yetter–Drinfeld modules k and V constructed in Example 2.7 will be denoted by ${}_f k^g$ and respectively ${}_f V^g$.

As an example, if $\alpha \in Aut_{Hopf}(H)$, then $(\varepsilon, 1)$ is a pair in involution corresponding to (α, α).

Let $\alpha, \beta \in Aut_{Hopf}(H)$. We define an H-bicomodule algebra $H(\alpha, \beta)$ as follows; $H(\alpha, \beta) = H$ as algebra, with comodule structures

$$H(\alpha, \beta) \to H \otimes H(\alpha, \beta), h \mapsto h_{[-1]} \otimes h_{[0]} = \alpha(h_1) \otimes h_2,$$
$$H(\alpha, \beta) \to H(\alpha, \beta) \otimes H, h \mapsto h_{<0>} \otimes h_{<1>} = h_1 \otimes \beta(h_2).$$

Then we can consider the Yetter–Drinfeld datum $(H, H(\alpha, \beta), H)$ and the Yetter–Drinfeld modules over it, ${}_{H(\alpha,\beta)}\mathcal{YD}(H)^H$.

Proposition 2.9: ${}_H\mathcal{YD}^H(\alpha, \beta) = {}_{H(\alpha,\beta)}\mathcal{YD}(H)^H$.

Proof: It is easy to see that the compatibility conditions for the two categories are the same. ∎

In particular, the category of anti-Yetter–Drinfeld modules coincides with ${}_{H(S^2,id)}\mathcal{YD}(H)^H$, which improves the remark in [7] that anti-Yetter–Drinfeld modules are entwined modules.

Consider now the diagonal crossed product $A(\alpha,\beta) = H^* \bowtie H(\alpha,\beta)$, whose multiplication is

$$(p \bowtie h)(q \bowtie l) = p(\alpha(h_1) \rightharpoonup q \leftharpoonup S^{-1}(\beta(h_3))) \bowtie h_2 l, \tag{2.6}$$

for all $p, q \in H^*$ and $h, l \in H$. For $\alpha = \beta = id$ we get $A(id, id) = D(H)$; for $\alpha = S^2$ and $\beta = id$, the multiplication in $A(S^2, id)$ is

$$(p \bowtie h)(q \bowtie l) = p(S^2(h_1) \rightharpoonup q \leftharpoonup S^{-1}(h_3)) \bowtie h_2 l, \tag{2.7}$$

hence $A(S^2, id)$ coincides with the algebra $A(H)$ defined in [7].

Assume now that H is finite dimensional; then $A(\alpha,\beta)$ becomes a $D(H)$-bicomodule algebra, with structures

$$H^* \bowtie H(\alpha,\beta) \to (H^* \bowtie H(\alpha,\beta)) \otimes D(H), p \bowtie h \mapsto (p_2 \bowtie h_1) \otimes (p_1 \otimes \beta(h_2)),$$
$$H^* \bowtie H(\alpha,\beta) \to D(H) \otimes (H^* \bowtie H(\alpha,\beta)), p \bowtie h \mapsto (p_2 \otimes \alpha(h_1)) \otimes (p_1 \bowtie h_2).$$

In particular, $A(H)$ becomes a $D(H)$-bicomodule algebra, improving the remark in [7] that $A(H)$ is a right $D(H)$-comodule algebra. Since H is finite dimensional, we have an isomorphism of categories ${}_{H(\alpha,\beta)}\mathcal{YD}(H)^H \simeq {}_{H^* \bowtie H(\alpha,\beta)}\mathcal{M}$, hence ${}_H\mathcal{YD}^H(\alpha,\beta) \simeq {}_{H^* \bowtie H(\alpha,\beta)}\mathcal{M}$ (for $\alpha = S^2$, $\beta = id$ we recover the result in [7] that the category of anti-Yetter–Drinfeld modules is isomorphic to ${}_{A(H)}\mathcal{M}$). The correspondence is given as follows. If $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, then $M \in {}_{H^* \bowtie H(\alpha,\beta)}\mathcal{M}$ with structure

$$(p \bowtie h) \cdot m = p((h \cdot m)_{(1)})(h \cdot m)_{(0)}.$$

Conversely, if $M \in_{H^* \bowtie H(\alpha,\beta)}\mathcal{M}$, then $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$ with structures

$$h \cdot m = (\varepsilon \bowtie h) \cdot m,$$
$$m \mapsto m_{(0)} \otimes m_{(1)} = (e^i \bowtie 1) \cdot m \otimes e_i,$$

where $\{e_i\}$, $\{e^i\}$ are dual bases in H and H^*.

3. A braided T-category $\mathcal{YD}(H)$

Let $\alpha, \beta \in Aut_{Hopf}(H)$ and consider the objects H_α, H_β as in Example 2.5. In [4] was considered the object $M = H_\alpha \otimes H_\beta$, with the following structures:

$$h \cdot (x \otimes y) = h_1 \cdot x \otimes \alpha(h_2) \cdot y,$$
$$x \otimes y \mapsto (x_1 \otimes y_1) \otimes y_2 x_2,$$

for all $h, x, y \in H$, where by $\cdot$ we denoted both the actions of H on H_α and H_β given as in Example 2.5. Then it was noted in [4] that M satisfies a compatibility condition which says that $M \in {}_H\mathcal{YD}^H(id, \beta\alpha)$.

On the other hand, it was noted in [7] that the tensor product between an anti-Yetter–Drinfeld module and a Yetter–Drinfeld module becomes an anti-Yetter–Drinfeld module.

The next result can be seen as a generalization of both these facts.

Proposition 3.1: *If $M \in {}_H\mathcal{YD}^H(\alpha, \beta)$, $N \in {}_H\mathcal{YD}^H(\gamma, \delta)$, then $M \otimes N \in {}_H\mathcal{YD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$, with structures*

$$\begin{aligned}
&h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n,\\
&m \otimes n \mapsto (m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} = (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)} m_{(1)}.
\end{aligned}$$

Proof: Obviously $M \otimes N$ is a left H-module and a right H-comodule. We check now the compatibility condition. We compute

$$\begin{aligned}
&(h \cdot (m \otimes n))_{(0)} \otimes (h \cdot (m \otimes n))_{(1)}\\
&\quad= \big(\gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n\big)_{(0)} \otimes \big(\gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n\big)_{(1)}\\
&\quad= \big((\gamma(h_1) \cdot m)_{(0)} \otimes (\gamma^{-1}\beta\gamma(h_2) \cdot n)_{(0)}\big) \otimes \big(\gamma^{-1}\beta\gamma(h_2) \cdot n\big)_{(1)} \big(\gamma(h_1) \cdot m\big)_{(1)}\\
&\quad= \big(\gamma(h_1)_2 \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma(h_2)_2 \cdot n_{(0)}\big)\\
&\qquad \otimes \delta(\gamma^{-1}\beta\gamma(h_2)_3) n_{(1)} \gamma(S^{-1}(\gamma^{-1}\beta\gamma(h_2)_1)) \beta(\gamma(h_1)_3) m_{(1)} \alpha(S^{-1}(\gamma(h_1)_1))\\
&\quad= \big(\gamma(h_2) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma(h_5) \cdot n_{(0)}\big)\\
&\qquad \otimes \delta\gamma^{-1}\beta\gamma(h_6) n_{(1)} \beta\gamma(S^{-1}(h_4)) \beta\gamma(h_3) m_{(1)} \alpha(S^{-1}(\gamma(h_1)))\\
&\quad= \big(\gamma(h_2) \cdot m_{(0)} \otimes \gamma^{-1}\beta\gamma(h_3) \cdot n_{(0)}\big) \otimes \delta\gamma^{-1}\beta\gamma(h_4) n_{(1)} m_{(1)} \alpha\gamma(S^{-1}(h_1))\\
&\quad= h_2 \cdot (m_{(0)} \otimes n_{(0)}) \otimes \delta\gamma^{-1}\beta\gamma(h_3) n_{(1)} m_{(1)} \alpha\gamma(S^{-1}(h_1))\\
&\quad= h_2 \cdot (m \otimes n)_{(0)} \otimes \delta\gamma^{-1}\beta\gamma(h_3) (m \otimes n)_{(1)} \alpha\gamma(S^{-1}(h_1)),
\end{aligned}$$

that is $M \otimes N \in {}_H\mathcal{YD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$. ∎

Note that, if $M \in {}_H\mathcal{YD}^H(\alpha, \beta)$, $N \in {}_H\mathcal{YD}^H(\gamma, \delta)$ and $P \in {}_H\mathcal{YD}^H(\mu, \nu)$, then $(M \otimes N) \otimes P = M \otimes (N \otimes P)$ as objects in ${}_H\mathcal{YD}^H(\alpha\gamma\mu, \nu\mu^{-1}\delta\gamma^{-1}\beta\gamma\mu)$.

Denote $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$, a group with multiplication

$$(\alpha,\beta) * (\gamma,\delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma) \tag{3.1}$$

(the unit is (id, id) and $(\alpha,\beta)^{-1} = (\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1})$).

Proposition 3.2: *Let $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $(\alpha,\beta) \in G$. Define ${}^{(\alpha,\beta)}N = N$ as vector space, with structures*

$$h \rightharpoonup n = \gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n,$$
$$n \mapsto n_{<0>} \otimes n_{<1>} = n_{(0)} \otimes \alpha\beta^{-1}(n_{(1)}).$$

Then

$${}^{(\alpha,\beta)}N \in {}_H\mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}) = {}_H\mathcal{YD}^H((\alpha,\beta) * (\gamma,\delta) * (\alpha,\beta)^{-1}).$$

Proof: Obviously ${}^{(\alpha,\beta)}N$ is a left H-module and right H-comodule, so we check the compatibility condition. We compute

$$\begin{aligned}
&(h \rightharpoonup n)_{<0>} \otimes (h \rightharpoonup n)_{<1>} \\
&\quad = (\gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n)_{(0)} \otimes \alpha\beta^{-1}((\gamma^{-1}\beta\gamma\alpha^{-1}(h) \cdot n)_{(1)}) \\
&\quad = \gamma^{-1}\beta\gamma\alpha^{-1}(h_2) \cdot n_{(0)} \otimes \alpha\beta^{-1}(\delta\gamma^{-1}\beta\gamma\alpha^{-1}(h_3)n_{(1)}\gamma\gamma^{-1}\beta\gamma\alpha^{-1}(S^{-1}(h_1))) \\
&\quad = \gamma^{-1}\beta\gamma\alpha^{-1}(h_2) \cdot n_{(0)} \otimes \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}(h_3)\alpha\beta^{-1}(n_{(1)})\alpha\gamma\alpha^{-1}(S^{-1}(h_1)) \\
&\quad = h_2 \rightharpoonup n_{(0)} \otimes \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}(h_3)n_{<1>}\alpha\gamma\alpha^{-1}(S^{-1}(h_1)),
\end{aligned}$$

that is ${}^{(\alpha,\beta)}N \in {}_H\mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})$. ∎

Remark 3.3: Let $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $(\mu,\nu) \in G$. Then we have

$${}^{(\alpha,\beta)*(\mu,\nu)}N = {}^{(\alpha,\beta)}({}^{(\mu,\nu)}N)$$

as objects in ${}_H\mathcal{YD}^H(\alpha\mu\gamma\mu^{-1}\alpha^{-1}, \alpha\beta^{-1}\mu\nu^{-1}\delta\gamma^{-1}\nu\mu^{-1}\beta\mu\gamma\mu^{-1}\alpha^{-1})$, and

$${}^{(\mu,\nu)}(M \otimes N) = {}^{(\mu,\nu)}M \otimes {}^{(\mu,\nu)}N$$

as objects in ${}_H\mathcal{YD}^H(\mu\alpha\gamma\mu^{-1}, \mu\nu^{-1}\delta\gamma^{-1}\beta\alpha^{-1}\nu\alpha\gamma\mu^{-1})$.

Proposition 3.4: *Let $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$ and $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$. Define ${}^MN = {}^{(\alpha,\beta)}N$ as object in ${}_H\mathcal{YD}^H((\alpha,\beta) * (\gamma,\delta) * (\alpha,\beta)^{-1})$. Define the map*

$$c_{M,N}\colon M \otimes N \to {}^MN \otimes M, c_{M,N}(m \otimes n) = n_{(0)} \otimes \beta^{-1}(n_{(1)}) \cdot m.$$

Then $c_{M,N}$ is H-linear H-colinear and satisfies the conditions:

$$c_{M\otimes N,P} = (c_{M,{}^N P} \otimes id_N) \circ (id_M \otimes c_{N,P}), \tag{3.2}$$

$$c_{M,N\otimes P} = (id_{{}^M N} \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P). \tag{3.3}$$

(for $P \in {}_H\mathcal{YD}^H(\mu,\nu)$). Moreover, if $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, $N \in {}_H\mathcal{YD}^H(\gamma,\delta)$ and $(\mu,\nu) \in G$, then $c_{{}^{(\mu,\nu)}M,{}^{(\mu,\nu)}N} = c_{M,N}$.

Proof: We prove that $c_{M,N}$ is H-linear. We compute

$$\begin{aligned}
&c_{M,N}(h\cdot(m\otimes n))\\
&\quad= c_{M,N}(\gamma(h_1)\cdot m \otimes \gamma^{-1}\beta\gamma(h_2)\cdot n)\\
&\quad= (\gamma^{-1}\beta\gamma(h_2)\cdot n)_{(0)} \otimes \beta^{-1}((\gamma^{-1}\beta\gamma(h_2)\cdot n)_{(1)})\gamma(h_1)\cdot m\\
&\quad= \gamma^{-1}\beta\gamma(h_2)_2\cdot n_{(0)}\\
&\qquad \otimes \beta^{-1}\big(\delta(\gamma^{-1}\beta\gamma(h_2)_3)n_{(1)}\gamma(S^{-1}((\gamma^{-1}\beta\gamma(h_2))_1))\big)\gamma(h_1)\cdot m\\
&\quad= \gamma^{-1}\beta\gamma(h_3)\cdot n_{(0)} \otimes \beta^{-1}\delta\gamma^{-1}\beta\gamma(h_4)\beta^{-1}(n_{(1)})\gamma(S^{-1}(h_2))\gamma(h_1)\cdot m\\
&\quad= \gamma^{-1}\beta\gamma(h_1)\cdot n_{(0)} \otimes \beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2)\beta^{-1}(n_{(1)})\cdot m,
\end{aligned}$$

$$\begin{aligned}
h\cdot c_{M,N}(m\otimes n) &= h\cdot(n_{(0)} \otimes \beta^{-1}(n_{(1)})\cdot m)\\
&= \alpha(h_1) \rightharpoonup n_{(0)} \otimes \alpha^{-1}\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1}\alpha(h_2)\beta^{-1}(n_{(1)})\cdot m\\
&= \gamma^{-1}\beta\gamma\alpha^{-1}\alpha(h_1)\cdot n_{(0)} \otimes \beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2)\beta^{-1}(n_{(1)})\cdot m\\
&= \gamma^{-1}\beta\gamma(h_1)\cdot n_{(0)} \otimes \beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2)\beta^{-1}(n_{(1)})\cdot m,
\end{aligned}$$

so the two terms are equal. The fact that $c_{M,N}$ is H-colinear is similar and left to the reader. We prove now (3.2). First note that, due to Remark 3.3, we have ${}^M({}^N P) = {}^{M\otimes N}P$ and ${}^M(N\otimes P) = {}^M N \otimes^M P$. We compute

$$\begin{aligned}
(c_{M,{}^N P} \otimes id_N)\circ(id_M \otimes c_{N,P})(m\otimes n\otimes p)&\\
&= c_{M,{}^N P}(m\otimes p_{(0)}) \otimes \delta^{-1}(p_{(1)})\cdot n\\
&= p_{(0)_{<0>}} \otimes \beta^{-1}(p_{(0)_{<1>}})\cdot m \otimes \delta^{-1}(p_{(1)})\cdot n\\
&= p_{(0)_{(0)}} \otimes \beta^{-1}\gamma\delta^{-1}(p_{(0)_{(1)}})\cdot m \otimes \delta^{-1}(p_{(1)})\cdot n\\
&= p_{(0)} \otimes \beta^{-1}\gamma\delta^{-1}(p_{(1)_1})\cdot m \otimes \delta^{-1}(p_{(1)_2})\cdot n,
\end{aligned}$$

$$\begin{aligned}
&c_{M\otimes N,P}(m\otimes n\otimes p)\\
&\quad=p_{(0)}\otimes\gamma^{-1}\beta^{-1}\gamma\delta^{-1}(p_{(1)})\cdot(m\otimes n)\\
&\quad=p_{(0)}\otimes\gamma\gamma^{-1}\beta^{-1}\gamma\delta^{-1}(p_{(1)_1})\cdot m\otimes\gamma^{-1}\beta\gamma\gamma^{-1}\beta^{-1}\gamma\delta^{-1}(p_{(1)_2})\cdot n\\
&\quad=p_{(0)}\otimes\beta^{-1}\gamma\delta^{-1}(p_{(1)_1})\cdot m\otimes\delta^{-1}(p_{(1)_2})\cdot n,
\end{aligned}$$

and we are done. The proof of (3.3) is easier and left to the reader, and similarly the last statement of the Proposition. ∎

Note that $c_{M,N}$ is bijective with inverse $c^{-1}_{M,N}(n\otimes m)=\beta^{-1}(S(n_{(1)}))\cdot m\otimes n_{(0)}$.

We are ready now to introduce the desired braided T-category (we use terminology as in [18]; for the subject of Turaev categories, see also the original paper of Turaev [16] and [2], [17]).

Define $\mathcal{YD}(H)$ as the disjoint union of all ${}_H\mathcal{YD}^H(\alpha,\beta)$, with $(\alpha,\beta)\in G$ (hence the component of the unit is just ${}_H\mathcal{YD}^H$). If we endow $\mathcal{YD}(H)$ with tensor product as in Proposition 3.1, then it becomes a strict monoidal category with a unit k as an object in ${}_H\mathcal{YD}^H$ (with trivial structures).

The group homomorphism $\varphi\colon G\to aut(\mathcal{YD}(H))$, $(\alpha,\beta)\mapsto\varphi_{(\alpha,\beta)}$, is given on components as

$$\varphi_{(\alpha,\beta)}\colon {}_H\mathcal{YD}^H(\gamma,\delta)\to{}_H\mathcal{YD}^H((\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}),\varphi_{(\alpha,\beta)}(N)={}^{(\alpha,\beta)}N,$$

and the functor $\varphi_{(\alpha,\beta)}$ acts as identity on morphisms. The braiding in $\mathcal{YD}(H)$ is given by the family $\{c_{M,N}\}$. As a consequence of the above results, we obtain

Theorem 3.5: *$\mathcal{YD}(H)$ is a braided T-category over G.*

We consider now the problem of existence of left and right dualities.

Proposition 3.6: *Let $M\in{}_H\mathcal{YD}^H(\alpha,\beta)$ and assume that M is finite dimensional. Then $M^*=\mathrm{Hom}(M,k)$ becomes an object in ${}_H\mathcal{YD}^H(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})$, with $(h\cdot f)(m)=f((\beta^{-1}\alpha^{-1}S(h))\cdot m)$ and $f_{(0)}(m)\otimes f_{(1)}=f(m_{(0)})\otimes S^{-1}(m_{(1)})$. Moreover, the maps $b_M\colon k\to M\otimes M^*$, $b_M(1)=\sum_i e_i\otimes e^i$ (where $\{e_i\}$ and $\{e^i\}$ are dual bases in M and M^*) and $d_M\colon M^*\otimes M\to k$, $d_M(f\otimes m)=f(m)$, are morphisms in ${}_H\mathcal{YD}^H$ and we have $(id_M\otimes d_M)(b_M\otimes id_M)=id_M$ and $(d_M\otimes id_{M^*})(id_{M^*}\otimes b_M)=id_{M^*}$.*

Proof: We first prove that M^* is indeed an object in ${}_H\mathcal{YD}^H(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})$. We compute:

$$\begin{aligned}
(h\cdot f)_{(0)}(m)\otimes(h\cdot f)_{(1)}&=(h\cdot f)(m_{(0)})\otimes S^{-1}(m_{(1)})\\
&=f((\beta^{-1}\alpha^{-1}S)(h)\cdot m_{(0)})\otimes S^{-1}(m_{(1)}),
\end{aligned}$$

$$\begin{aligned}
&(h_{(2)}\cdot f_{(0)})(m)\otimes(\alpha\beta^{-1}\alpha^{-1})(h_{(3)})f_{(1)}(\alpha^{-1}S^{-1})(h_{(1)})\\
&\quad=f_{(0)}\big(\beta^{-1}\alpha^{-1}S(h_{(2)})\cdot m\big)\otimes(\alpha\beta^{-1}\alpha^{-1})(h_{(3)})f_{(1)}(\alpha^{-1}S^{-1}(h_{(1)}))\\
&\quad=f\big(((\beta^{-1}\alpha^{-1}S)(h_{(2)})\cdot m)_{(0)}\big)\otimes(\alpha\beta^{-1}\alpha^{-1}(h_{(3)}))\\
&\qquad\times S^{-1}\big(((\beta^{-1}\alpha^{-1}S)(h_{(2)})\cdot m)_{(1)}\big)(\alpha^{-1}S^{-1}(h_{(1)}))\\
&\quad=f\big((\beta^{-1}\alpha^{-1}S)(h_{(3)})\cdot m_{(0)}\big)\otimes(\alpha\beta^{-1}\alpha^{-1}(h_{(5)}))\\
&\qquad\times S^{-1}\big((\alpha^{-1}S)(h_{(2)})m_{(1)}(\alpha\beta^{-1}\alpha^{-1})(h_{(4)})\big)(\alpha^{-1}S^{-1}(h_{(1)}))\\
&\quad=f\big((\beta^{-1}\alpha^{-1}S)(h_{(3)})\cdot m_{(0)}\big)\otimes(\alpha\beta^{-1}\alpha^{-1})(h_{(5)}S^{-1}(h_{(4)}))S^{-1}(m_{(1)})\\
&\qquad\times\alpha^{-1}\big(h_{(2)}S^{-1}(h_{(1)})\big)\\
&\quad=f\big((\beta^{-1}\alpha^{-1}S)(h)\cdot m_{(0)}\big)\otimes S^{-1}(m_{(1)}),
\end{aligned}$$

which means that

$$(h\cdot f)_{(0)}\otimes(h\cdot f)_{(1)}=(h_{(2)}\cdot f_{(0)})\otimes(\alpha\beta^{-1}\alpha^{-1})(h_{(3)})f_{(1)}(\alpha^{-1}S^{-1})(h_{(1)}). \qquad \blacksquare$$

On k we have the trivial module and comodule structure, and with these $k\in{}_H\mathcal{YD}^H$. We want to prove that b_M and d_M are H-module maps. We compute:

$$\begin{aligned}
(h\cdot b_M(1))(m)&=(h\cdot(\sum_i e_i\otimes e^i))(m)\\
&=\sum_i\alpha^{-1}(h_{(1)})\cdot e_i\otimes((\alpha\beta\alpha^{-1})(h_{(2)})\cdot e^i)(m)\\
&=\alpha^{-1}(h_{(1)})\cdot e_i\otimes e^i((\beta^{-1}\alpha^{-1}S\alpha\beta\alpha^{-1})(h_{(2)})\cdot m)\\
&=\sum_i\alpha^{-1}(h_{(1)})\cdot e_i\otimes e^i((\alpha^{-1}S)(h_{(2)})\cdot m)\\
&=\alpha^{-1}(h_{(1)}S(h_{(2)}))\cdot m\\
&=\varepsilon(h)\sum_i e_i\otimes e^i(m)\\
&=(\varepsilon(h)b_M(1))(m),\\
d_M(h\cdot(f\otimes m))&=d_M(\alpha(h_{(1)})\cdot f\otimes\beta^{-1}(h_{(2)})\cdot m)\\
&=(\alpha(h_{(1)})\cdot f)(\beta^{-1}(h_{(2)})\cdot m)\\
&=f((\beta^{-1}\alpha^{-1}S\alpha(h_{(1)}))\beta^{-1}(h_{(2)})\cdot m)\\
&=f(\beta^{-1}(S(h_{(1)})h_{(2)})\cdot m)\\
&=\varepsilon(h)d_M(f\otimes m).
\end{aligned}$$

They also are H-comodule maps;

$$\begin{aligned}((b_M(1))_{(0)}\otimes(b_M(1))_{(1)})(m) &= \sum_i (e_i)_{(0)}\otimes(e^i)_{(0)}(m)\otimes(e^i)_{(1)}(e_i)_{(1)}\\ &= \sum_i (e_i)_{(0)}\otimes(e^i)(m_{(0)})\otimes S^{-1}(m_{(1)})(e_i)_{(1)}\\ &= m_{(0)}\otimes S^{-1}(m_{(1)_2})m_{(1)_1}\\ &= (b_M(1)\otimes 1)(m),\\ d_M((f\otimes m)_{(0)})\otimes(f\otimes m)_{(1)} &= f_{(0)}(m_{(0)})\otimes m_{(1)}f_{(1)}\\ &= f(m_{(0)})\otimes m_{(1)_2}S^{-1}(m_{(1)_1})\\ &= d_M(f\otimes m)\otimes 1.\end{aligned}$$

Finally, the last two identities

$$(id_M\otimes d_M)(b_M\otimes id_M)=id_M \quad\text{and}\quad (d_M\otimes id_{M^*})(id_{M^*}\otimes b_M)=id_{M^*}$$

are trivial. ∎

Similarly, one can prove

PROPOSITION 3.7: *Let $M\in {}_H\mathcal{YD}^H(\alpha,\beta)$ and assume that M is finite dimensional. Then ${}^*M=\mathrm{Hom}(M,k)$ becomes an object in ${}_H\mathcal{YD}^H(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})$, with $(h\cdot f)(m)=f((\beta^{-1}\alpha^{-1}S^{-1}(h))\cdot m)$ and $f_{(0)}(m)\otimes f_{(1)}=f(m_{(0)})\otimes S(m_{(1)})$. Moreover, the maps $b_M\colon k\to{}^*M\otimes M$, $b_M(1)=\sum_i e^i\otimes e_i$ and $d_M\colon M\otimes{}^*M\to k$, $d_M(m\otimes f)=f(m)$, are morphisms in ${}_H\mathcal{YD}^H$ and we have*

$$(d_M\otimes id_M)(id_M\otimes b_M)=id_M \quad\text{and}\quad (id_{^*M}\otimes d_M)(b_M\otimes id_{^*M})=id_{^*M}.$$

Consequently, if we consider $\mathcal{YD}(H)_{fd}$, the subcategory of $\mathcal{YD}(H)$ consisting of finite dimensional objects, we obtain

THEOREM 3.8: *$\mathcal{YD}(H)_{fd}$ is a braided T-category with left and right dualities over G, the left (respectively right) duals being given as in Proposition 3.6 (respectively Proposition 3.7).*

Assume now that H is finite dimensional. We will construct a quasitriangular T-coalgebra over G, denoted by $DT(H)$, with the property that the T-category $Rep(DT(H))$ of representations of $DT(H)$ is isomorphic to $\mathcal{YD}(H)$ as braided T-categories.

For $(\alpha, \beta) \in G$, the (α, β)-component $DT(H)_{(\alpha,\beta)}$ will be the diagonal crossed product algebra $H^* \bowtie H(\alpha, \beta)$. Define

$$\Delta_{(\alpha,\beta),(\gamma,\delta)}: H^* \bowtie H((\alpha, \beta) * (\gamma, \delta)) \to (H^* \bowtie H(\alpha, \beta)) \otimes (H^* \bowtie H(\gamma, \delta)),$$
$$\Delta_{(\alpha,\beta),(\gamma,\delta)}(p \bowtie h) = (p_2 \bowtie \gamma(h_1)) \otimes (p_1 \bowtie \gamma^{-1}\beta\gamma(h_2)).$$

One can check, by direct computation, that these maps are algebra maps, satisfying the necessary coassociativity conditions.

The counit ε is just the counit of $DT(H)_{(id,id)} = D(H)$, the Drinfeld double of H.

For $(\alpha, \beta), (\gamma, \delta) \in G$, define now

$$\varphi_{(\alpha,\beta)}^{(\gamma,\delta)}: H^* \bowtie H(\gamma, \delta) \to H^* \bowtie H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}),$$
$$\varphi_{(\alpha,\beta)}^{(\gamma,\delta)}(p \bowtie h) = p \circ \beta\alpha^{-1} \bowtie \alpha\gamma^{-1}\beta^{-1}\gamma(h).$$

Then one can check by direct computation that these are algebra isomorphisms giving a *conjugation* (that is they are multiplicative and compatible with the comultiplications and the counit).

The antipode is given, for $(\alpha, \beta) \in G$, by

$$S_{(\alpha,\beta)}: H^* \bowtie H(\alpha, \beta) \to H^* \bowtie H((\alpha, \beta)^{-1}),$$
$$S_{(\alpha,\beta)}(p \bowtie h) = (\varepsilon \bowtie \alpha\beta(S(h))) \cdot (S^{*-1}(p) \bowtie 1),$$

where the multiplication $\cdot$ in the right hand side is made in $H^* \bowtie H((\alpha, \beta)^{-1})$.

Finally, the universal R-matrix is given by

$$R_{(\alpha,\beta),(\gamma,\delta)} = \sum_i (\varepsilon \bowtie \beta^{-1}(e_i)) \otimes (e^i \bowtie 1) \in (H^* \bowtie H(\alpha, \beta)) \otimes (H^* \bowtie H(\gamma, \delta)),$$

for all $(\alpha, \beta), (\gamma, \delta) \in G$, where $\{e_i\}$, $\{e^i\}$ are dual bases in H and H^*.

Thus, we have obtained

THEOREM 3.9: *$DT(H)$ is a quasitriangular T-coalgebra over G, with structure as above.*

Moreover, the structure of $DT(H)$ was constructed in such a way that, via the isomorphisms ${}_{H^* \bowtie H(\alpha,\beta)}\mathcal{M} \simeq {}_H\mathcal{YD}^H(\alpha, \beta)$ from Section 2, we obtain

THEOREM 3.10: *$Rep(DT(H))$ and $\mathcal{YD}(H)$ are isomorphic as braided T-categories over G.*

Remark 3.11: From $\mathcal{YD}(H)$ (respectively $DT(H)$) we can obtain, by pull-back along the group morphism $Aut_{Hopf}(H) \to G$, $\alpha \mapsto (\alpha, id)$, a braided T-category (respectively a quasitriangular T-coalgebra) over $Aut_{Hopf}(H)$. If π is a group together with a group morphism $\pi \to Aut_{Hopf}(H)$, by pull-back along it we obtain a braided T-category (respectively a quasitriangular T-coalgebra) over π. Quasitriangular T-coalgebras over such π have been studied by Virelizier in [17].

4. The case of group algebras

The aim of this section is to describe $\mathcal{YD}(H)$ if H is the group algebra $k[A]$ of a group A.

First, it is well-known that $Aut_{Hopf}(k[A]) = Aut(A)$, the group of automorphisms of A. Let then $\alpha, \beta \in Aut(A)$; an (α, β)-Yetter–Drinfeld module over $k[A]$ is a left A-module M with a decomposition $M = \bigoplus_{a\in A} M_a$ such that, for all $a, b \in A$, we have $a \cdot M_b \subseteq M_{\beta(a)b\alpha(a^{-1})}$.

If $\alpha, \beta, \gamma, \delta \in Aut(A)$, $M = \bigoplus_{a\in A} M_a \in {}_{k[A]}\mathcal{YD}^{k[A]}(\alpha, \beta)$ and $N = \bigoplus_{a\in A} N_a \in {}_{k[A]}\mathcal{YD}^{k[A]}(\gamma, \delta)$, then $M \otimes N \in {}_{k[A]}\mathcal{YD}^{k[A]}(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$ with action $a \cdot (m \otimes n) = \gamma(a) \cdot m \otimes \gamma^{-1}\beta\gamma(a) \cdot n$, for all $a \in A$, $m \in M$, $n \in N$, and decomposition $M \otimes N = \bigoplus_{c\in A}(\bigoplus_{ba=c} M_a \otimes N_b)$.

If $\alpha, \beta \in Aut(A)$ and $N = \bigoplus_{a\in A} N_a \in {}_{k[A]}\mathcal{YD}^{k[A]}(\gamma, \delta)$, then ${}^{(\alpha,\beta)}N = N$ as vector space, with action $a \rightharpoonup n = \gamma^{-1}\beta\gamma\alpha^{-1}(a) \cdot n$, for all $a \in A$, $n \in N$, and decomposition ${}^{(\alpha,\beta)}N = \bigoplus_{a\in A} N_{\beta\alpha^{-1}(a)}$.

Let

$$M = \bigoplus_{a\in A} M_a \in {}_{k[A]}\mathcal{YD}^{k[A]}(\alpha, \beta) \quad \text{and} \quad N = \bigoplus_{b\in A} N_b \in {}_{k[A]}\mathcal{YD}^{k[A]}(\gamma, \delta);$$

then the braiding $c_{M,N}\colon M \otimes N \to {}^{M}N \otimes M$ acts on homogeneous elements $m \in M_a$ and $n \in N_b$ as $c_{M,N}(m \otimes n) = n \otimes \beta^{-1}(b) \cdot m$, hence it sends $M_a \otimes N_b$ to $N_b \otimes M_{ba\alpha\beta^{-1}(b^{-1})}$.

Let now $M = \bigoplus_{a\in A} M_a \in {}_{k[A]}\mathcal{YD}^{k[A]}(\alpha, \beta)$ finite dimensional. Since $S = S^{-1}$ for $k[A]$, we have that $M^* = {}^*M$, and it can be described as follows: the action is $(a \cdot f)(m) = f(\beta^{-1}\alpha^{-1}(a^{-1}) \cdot m)$, for all $a \in A$, $f \in M^*$, $m \in M$, and the decomposition is $M^* = \bigoplus_{a\in A}(M_{a^{-1}})^*$.

Assume now that the group A is finite; we describe the structure of the quasitriangular T-coalgebra $DT(k[A])$. We denote by $\{p_a\}_{a\in A}$ the basis of the dual Hopf algebra $k[A]^*$, given by $p_a(b) = \delta_{a,b}$ for all $b \in A$. If $\alpha, \beta \in Aut(A)$, the component $DT(k[A])_{(\alpha,\beta)}$ is $k[A]^*\otimes k[A]$ as vector space, with multiplication

$$(p_a \otimes b)(p_c \otimes d) = \delta_{a\alpha(b),\beta(b)c} p_a \otimes bd,$$

for all $a, b, c, d \in A$; the unit is $1 = \sum_{a\in A} p_a \otimes 1$.

The structure maps of $DT(k[A])$ are given, for all $\alpha, \beta, \gamma, \delta \in Aut(A)$ and $a, b \in A$, by

$$\begin{aligned}
&\Delta_{(\alpha,\beta),(\gamma,\delta)}(p_a \otimes b) = \sum_{cd=a} (p_d \otimes \gamma(b)) \otimes (p_c \otimes \gamma^{-1}\beta\gamma(b)),\\
&\varepsilon(p_a \otimes b) = \delta_{a,1},\\
&\varphi^{(\gamma,\delta)}_{(\alpha,\beta)}(p_a \otimes b) = p_{\alpha\beta^{-1}(a)} \otimes \alpha\gamma^{-1}\beta^{-1}\gamma(b),\\
&S_{(\alpha,\beta)}(p_a \otimes b) = p_{\alpha(b^{-1})a^{-1}\beta(b)} \otimes \alpha\beta(b^{-1}),\\
&R_{(\alpha,\beta),(\gamma,\delta)} = \sum_{a,b\in A} (p_a \otimes \beta^{-1}(b)) \otimes (p_b \otimes 1).
\end{aligned}$$

5. An isomorphism of categories ${}_H\mathcal{YD}^H(\alpha,\beta) \simeq {}_H\mathcal{YD}^H$ in the presence of a pair in involution

The aim of this section is to prove the following result.

THEOREM 5.1: *Let $\alpha, \beta \in Aut_{Hopf}(H)$ and assume that there exists (f, g) a pair in involution corresponding to (α, β). Then the categories ${}_H\mathcal{YD}^H(\alpha,\beta)$ and ${}_H\mathcal{YD}^H$ are isomorphic.*

A pair of inverse functors (F, G) is given as follows. If $M \in {}_H\mathcal{YD}^H(\alpha,\beta)$, then $F(M) \in {}_H\mathcal{YD}^H$, where $F(M) = M$ as vector space, with structures

$$\begin{aligned}
&h \to m = f(\beta^{-1}(S(h_1)))\beta^{-1}(h_2) \cdot m,\\
&m \mapsto m_{<0>} \otimes m_{<1>} = m_{(0)} \otimes m_{(1)}g^{-1}.
\end{aligned}$$

If $N \in {}_H\mathcal{YD}^H$, then $G(N) \in {}_H\mathcal{YD}^H(\alpha,\beta)$, where $G(N) = N$ as vector space, with structures

$$\begin{aligned}
&h \rightharpoonup n = f(h_1)\beta(h_2) \cdot n,\\
&n \mapsto n^{(0)} \otimes n^{(1)} = n_{(0)} \otimes n_{(1)}g.
\end{aligned}$$

Both F and G act as identities on morphisms.

Proof: One checks, by direct computation, that F and G are functors, inverse to each other.

Alternatively, we can give a very short proof using results from Section 3. By Example 2.7, we have ${}_fk^g \in {}_H\mathcal{YD}^H(\alpha,\beta)$. By Proposition 3.6, we get $({}_fk^g)^* \in {}_H\mathcal{YD}^H((\alpha,\beta)^{-1})$. Then, one can check that actually $F(M) = ({}_fk^g)^* \otimes M \in_H \mathcal{YD}^H$ and $G(N) = ({}_fk^g) \otimes N \in_H \mathcal{YD}^H(\alpha,\beta)$. Also, one can see that $({}_fk^g)^* \otimes {}_fk^g = {}_fk^g \otimes ({}_fk^g)^* = k$ as objects in ${}_H\mathcal{YD}^H$, hence $F \circ G = G \circ F = id$, using the associativity of the tensor product. ∎

As we have noted before, for any $\alpha \in Aut_{Hopf}(H)$ we have that $(\varepsilon, 1)$ is a pair in involution corresponding to (α,α), hence we obtain

Corollary 5.2: ${}_H\mathcal{YD}^H(\alpha,\alpha) \simeq {}_H\mathcal{YD}^H$.

Also, as a consequence of the theorem, we obtain the following result (a right-left version was given in [14]), which might be useful for the area of applicability of anti-Yetter-Drinfeld modules:

Corollary 5.3: *Assume that there exists a pair in involution (f,g) corresponding to (S^2, id). Then the category ${}_H\mathcal{YD}^H(S^2,id)$ of anti-Yetter-Drinfeld modules is isomorphic to ${}_H\mathcal{YD}^H$, and any anti-Yetter-Drinfeld module can be written as a tensor product ${}_fk^g \otimes N$, with $N \in_H \mathcal{YD}^H$.*

Let again $\alpha, \beta \in Aut_{Hopf}(H)$ such that there exists (f,g) a pair in involution corresponding to (α,β), and assume that H is finite dimensional. Then we know that ${}_H\mathcal{YD}^H(\alpha,\beta) \simeq {}_{H^*\bowtie H(\alpha,\beta)}\mathcal{M}$, ${}_H\mathcal{YD}^H \simeq {}_{D(H)}\mathcal{M}$, and the isomorphism ${}_H\mathcal{YD}^H(\alpha,\beta) \simeq {}_H\mathcal{YD}^H$ constructed in the theorem is induced by an algebra isomorphism between $H^* \bowtie H(\alpha,\beta)$ and $D(H)$, given by

$$D(H) \to H^* \bowtie H(\alpha,\beta), p \otimes h \mapsto g^{-1} \rightharpoonup p \bowtie f(\beta^{-1}(S(h_1)))\beta^{-1}(h_2),$$
$$H^* \bowtie H(\alpha,\beta) \to D(H), p \bowtie h \mapsto g \rightharpoonup p \otimes f(h_1)\beta(h_2).$$

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